# Exercise 6

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' - 4y' + 4y = x - \sin x$$

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**Solution**

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 4y_c' + 4y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - 4(re^{rx}) + 4(e^{rx}) = 0$$

Divide both sides by $e^{rx}$.

$$r^2 - 4r + 4 = 0$$

Solve for $r$.

$$(r-2)^2 = 0$$

$$r = \{2\}$$

Two solutions to the ODE are $e^{2x}$ and $xe^{2x}$. By the principle of superposition, then,

$$y_c(x) = C_1 e^{2x} + C_2 x e^{2x}.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - 4y_p' + 4y_p = x - \sin x \tag{2}$$

Since the inhomogeneous term is the sum of a polynomial of degree 1 and a sine, the particular solution is $y_p = Ax + B + C\cos x + D\sin x$.

$$y_p = Ax + B + C\cos x + D\sin x \quad \rightarrow \quad y_p' = A - C\sin x + D\cos x \quad \rightarrow \quad y_p'' = -C\cos x - D\sin x$$

Substitute these formulas into equation (2).

$$(-C\cos x - D\sin x) - 4(A - C\sin x + D\cos x) + 4(Ax + B + C\cos x + D\sin x) = x - \sin x$$

$$(-4A + 4B) + (4A)x + (-C - 4D + 4C)\cos x + (-D + 4C + 4D)\sin x = x - \sin x$$

Match the coefficients on both sides to get a system of equations for $A$, $B$, $C$, and $D$.

$$\left.\begin{aligned} -4A + 4B &= 0 \\ 4A &= 1 \\ -C - 4D + 4C &= 0 \\ -D + 4C + 4D &= -1 \end{aligned}\right\}$$

Solving it yields

$$A = \frac{1}{4} \quad \text{and} \quad B = \frac{1}{4} \quad \text{and} \quad C = -\frac{4}{25} \quad \text{and} \quad D = -\frac{3}{25}.$$

which means the particular solution is

$$y_p = \frac{1}{4}x + \frac{1}{4} - \frac{4}{25}\cos x - \frac{3}{25}\sin x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{4}x + \frac{1}{4} - \frac{4}{25}\cos x - \frac{3}{25}\sin x, \end{aligned}$$

where $C_1$ and $C_2$ are arbitrary constants.